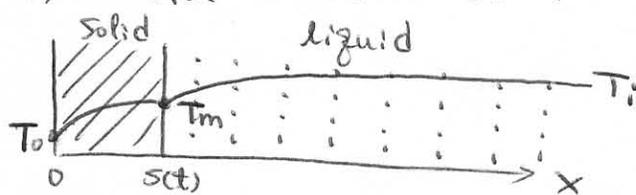


6.2.2. The Exact Solution Method.

* The solidification problem (1D)

A liquid at a uniform temperature T_i ($> T_m$) is confined to a half space $x > 0$. At $t=0$ the boundary surface at $x=0$ is lowered to T_0 ($< T_m$) and maintained for $t > 0$.



Solidification starts at $x=0$
Solid-liquid interface moves in $+x$ direction.

① For the solid phase:

$$\frac{\partial^2 T_s}{\partial x^2} = \frac{1}{\alpha_s} \frac{\partial T_s}{\partial t} \quad \alpha < x < s(t)$$

B.C. $T_s|_{x=0} = T_0$

We choose: $T_s(x,t) = T_0 + A \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha_s t}}\right)$ (A: unknown coefficient)

which satisfies the differential equation and the B.C.

[Note: $\operatorname{erf}\left(\frac{x}{\sqrt{4\alpha_s t}}\right)$ is the solution of the equation with $T_s = 0$ at $x=0$.]

② For the liquid phase

$$\frac{\partial^2 T_l}{\partial x^2} = \frac{1}{\alpha_l} \frac{\partial T_l}{\partial t} \quad s(t) < x < \infty$$

B.C. $T_l|_{x=\infty} = T_i$
 I.C. $T_l|_{t=0} = T_i$

We choose: $T_l(x, t) = T_i + B \operatorname{erfc}\left(\frac{x}{\sqrt{4\alpha_l t}}\right)$ (B: unknown coefficient)

Which satisfies the differential equation and the B.C./I.C.

[Note: $\operatorname{erfc}\left(\frac{x}{\sqrt{4\alpha_l t}}\right)$ is the solution of the equation with $T_l = 0$ at $x = \infty / t = 0$.]

③ At the interface:

Applying $T_s(x, t)|_{s(t)} = T_l(x, t)|_{s(t)} = T_m$

so: $T_0 + A \operatorname{erf}\left(\frac{s(t)}{\sqrt{4\alpha_s t}}\right) = T_i + B \operatorname{erfc}\left(\frac{s(t)}{\sqrt{4\alpha_l t}}\right) = T_m$

(two unknown plus s(t))

define: $\lambda \equiv \frac{s(t)}{\sqrt{4\alpha_s t}}$ [i.e. $s(t) = \lambda \sqrt{4\alpha_s t}$]

so: $T_0 + A \operatorname{erf}(\lambda) = T_i + B \operatorname{erfc}\left(\lambda \cdot \left(\frac{\alpha_s}{\alpha_l}\right)^{\frac{1}{2}}\right) = T_m \leftarrow \text{constant}$

therefore: λ is a constant (to be determined)

and:
$$\begin{cases} A = \frac{T_m - T_0}{\operatorname{erf}(\lambda)} \\ B = \frac{T_m - T_i}{\operatorname{erfc}\left(\lambda \sqrt{\frac{\alpha_s}{\alpha_l}}\right)} \end{cases}$$

$$\text{So: } \begin{cases} T_s(x,t) = T_0 + \frac{T_m - T_0}{\text{erf}(\lambda)} \text{erf}\left(\frac{x}{\sqrt{4\alpha_s t}}\right) \\ T_e(x,t) = T_i + \frac{T_m - T_i}{\text{erfc}(\lambda\sqrt{\alpha_s/\alpha_e})} \text{erfc}\left(\frac{x}{\sqrt{4\alpha_s t}}\right) \end{cases}$$

$$\text{Applying } \left[k_s \frac{\partial T_s}{\partial x} \Big|_{s(t)} - k_e \frac{\partial T_e}{\partial x} \Big|_{s(t)} = \rho_s L \frac{ds(t)}{dt} \right]$$

$$k_s \cdot \frac{T_m - T_0}{\text{erf}(\lambda)} \cdot \frac{1}{\sqrt{4\alpha_s t}} \cdot \frac{2}{\sqrt{\pi}} e^{-\frac{x^2}{4\alpha_s t}} \Big|_{s(t)} - k_e \frac{T_m - T_i}{\text{erfc}(\lambda\sqrt{\alpha_s/\alpha_e})} \cdot \frac{-1}{\sqrt{4\alpha_s t}} \cdot \frac{2}{\sqrt{\pi}} e^{-\frac{x^2}{4\alpha_s t}} \Big|_{s(t)}$$

$$= \rho_s L \cdot 2\lambda\sqrt{\alpha_s} \cdot \frac{1}{2\sqrt{t}}$$

$$\frac{k_s}{\sqrt{4\alpha_s t}} \cdot \frac{T_m - T_0}{\text{erf}(\lambda)} \cdot \frac{2}{\sqrt{\pi}} e^{-\lambda^2} + \frac{k_e}{\sqrt{4\alpha_s t}} \cdot \frac{T_m - T_i}{\text{erfc}(\lambda\sqrt{\alpha_s/\alpha_e})} \cdot \frac{2}{\sqrt{\pi}} e^{-\lambda^2 \frac{\alpha_s}{\alpha_e}} = \frac{\rho_s L \sqrt{\alpha_s}}{\sqrt{t}}$$

$$\frac{e^{-\lambda^2}}{\text{erf}(\lambda)} + \frac{k_e}{k_s} \sqrt{\frac{\alpha_s}{\alpha_e}} \cdot \frac{T_m - T_i}{T_m - T_0} \cdot \frac{e^{-\lambda^2 \frac{\alpha_s}{\alpha_e}}}{\text{erfc}(\lambda\sqrt{\alpha_s/\alpha_e})} = \frac{\rho_s L \sqrt{\alpha_s} \sqrt{\pi}}{k_s (T_m - T_0)}$$

$$\alpha_s = \frac{k_s}{\rho_s C_{ps}}$$

$$\text{i.e., } \frac{e^{-\lambda^2}}{\text{erf}(\lambda)} + \frac{k_e}{k_s} \sqrt{\frac{\alpha_s}{\alpha_e}} \cdot \frac{T_m - T_i}{T_m - T_0} \cdot \frac{e^{-\lambda^2 \frac{\alpha_s}{\alpha_e}}}{\text{erfc}(\lambda\sqrt{\alpha_s/\alpha_e})} = \frac{\lambda L \sqrt{\pi}}{C_{ps} (T_m - T_0)}$$

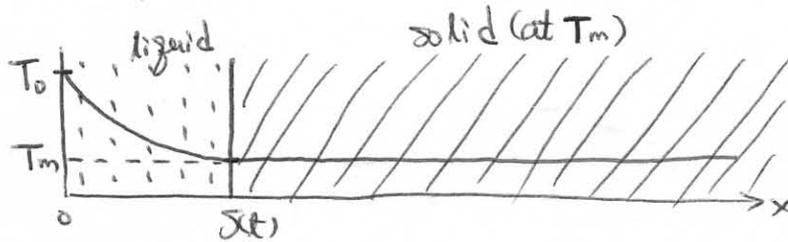
The equation for λ !

Once λ is determined, $s(t) = \lambda\sqrt{4\alpha_s t}$, and $T_s(x,t)$ and $T_e(x,t)$ are determined.

6.3 Integral Method (Approximate method)

* Example.

Consider melting of a solid confined to a half space $x > 0$, initially at the melting temperature T_m . For $t > 0$ the boundary surface at $x=0$ is kept at a constant temperature $T_0 (> T_m)$.



The complete problem.

For liquid phase.

$$\frac{\partial^2 T_l}{\partial x^2} = \frac{1}{\alpha_l} \frac{\partial T_l}{\partial t} \quad \alpha x < s(t)$$

B.C. $T_l|_{x=0} = T_0$

Interface condition:

$$T_l|_{x=s(t)} = T_m$$

$$-k_l \frac{\partial T_l}{\partial x} \Big|_{s(t)} = \rho L \frac{ds(t)}{dt}$$

For solid phase.

$$T_s = T_m$$

Step 1. Define a thermal layer $\delta(t)$, such that.

$$\begin{cases} T_s|_{x=\delta(t)} = T_m & \text{(unaffected)} \end{cases}$$

$$\begin{cases} \frac{\partial T_s}{\partial x} \Big|_{x=\delta(t)} = 0 & \text{in the solid!} \end{cases}$$

Therefore, $\delta(t) = s(t)$ (identical to the thermal layer definition)

Step 2: Integrate original equation from $0 \rightarrow s(t)$:

$$\int_0^{s(t)} \frac{\partial T_e}{\partial x^2} dx = \int_0^{s(t)} \frac{1}{\alpha_1} \frac{\partial T_e}{\partial t} dx$$

$$\frac{\partial T_e}{\partial x} \Big|_{x=s(t)} - \frac{\partial T_e}{\partial x} \Big|_{x=0} = \frac{1}{\alpha_1} \left[\frac{d}{dt} \int_{x=0}^{s(t)} T_e dx - \cancel{T_e} \Big|_{x=s(t)} \frac{ds(t)}{dt} \right]$$

(Interface condition)

$$\underline{-\frac{\rho L}{k_e} \frac{ds(t)}{dt} - \frac{\partial T_e}{\partial x} \Big|_{x=0} = \frac{1}{\alpha_1} \frac{d}{dt} \left[\int_{x=0}^{s(t)} T_e dx - T_m s(t) \right]}$$

energy
integral
equation

Step 3: choose an appropriate temperature distribution:

$$T_e(x, t) = a(t) + b(t)(x - s(t)) + c(t)(x - s(t))^2$$

Three conditions are needed to determine coefficients.

$$\left\{ \begin{array}{l} \checkmark T_e|_{x=0} = T_0 \\ \checkmark T_e|_{x=s(t)} = T_m \end{array} \right. \quad (\text{Note: } \frac{\partial T_e}{\partial t} \Big|_{x=s(t)} \neq 0)$$

$$\frac{\partial T_e}{\partial x} \Big|_{x=s(t)} = -\frac{\rho L}{k_e} \frac{ds(t)}{dt}$$

$$\text{or: } \left(\frac{\partial T_e}{\partial x} \right)^2 = \frac{\alpha_1 \rho L}{k_e} \frac{\partial^2 T_e}{\partial x^2} \Leftrightarrow \left\{ \begin{array}{l} \frac{dT_e}{dx} \Big|_{x=s(t)} = 0 \Rightarrow \frac{\partial T_e}{\partial x} \Big|_{s(t)} \cdot \frac{ds(t)}{dt} = -\frac{\partial T_e}{\partial t} \Big|_{s(t)} \\ \frac{\partial T_e}{\partial x} \Big|_{x=s(t)} = -\frac{\rho L}{k_e} \frac{ds(t)}{dt} \\ \frac{\partial T_e}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial T_e}{\partial t} \end{array} \right.$$

$$\text{So: } \left\{ \begin{array}{l} a(t) = T_m \\ b(t) = \frac{\alpha_1 \rho L}{k_e s(t)} [1 - (1 + \mu)^{1/2}] \\ c(t) = \frac{b(t) \cdot s(t) + (T_0 - T_m)}{s(t)^2} \end{array} \right.$$

$$\text{with } \mu \equiv \frac{2k_e}{\alpha_1 \rho L} (T_0 - T_m)$$

Step ④: Introduce $T_e(x,t)$ into "energy integral equation":

$$\text{We have: } S(t) \frac{ds(t)}{dt} = 6\alpha_e \frac{1 - (H\mu)^{1/2} + \mu}{5 + (H\mu)^{1/2} + \mu}$$

$$\text{I.C. } S(t) \Big|_{t=0} = 0$$

$$\text{So: } \boxed{S(t) = 2\lambda \sqrt{\rho t}}$$

$$\text{with } \lambda = \sqrt{3 \frac{1 - (H\mu)^{1/2} + \mu}{5 + (H\mu)^{1/2} + \mu}}$$

$$\text{Therefore: } \underline{T_e(x,t) = T_m + \frac{\alpha_e \rho L [1 - (H\mu)^{1/2}] (x - \sqrt{\rho t})}{K_e} + \frac{\alpha_e \rho L [1 - (H\mu)^{1/2}] + K_e (T_0 - T_m) (x - \sqrt{\rho t})^2}{K_e}}$$